Maximum Clique Interdiction Problems

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ESR Days

Outline

Clique Interdiction Problems

Structural Properties, Modeling and Exact Algorithms

Computational Results

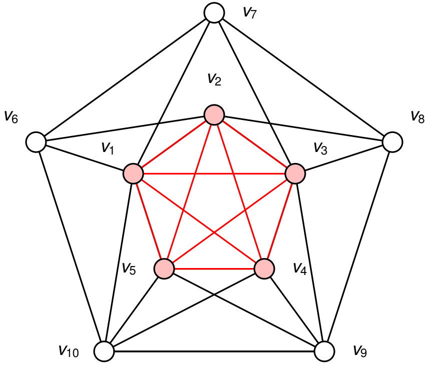
Facial study

References

[1] F. F., I. Ljubić, P. San Segundo, and S. Martin.
The maximum clique interdiction problem.
European Journal of Operational Research, 277(1), 112-127, 2019.

[2] F. F., I. Ljubić, P. San Segundo, and Y. Zhao.
A branch-and-cut algorithm for the Edge Interdiction Clique Problem.
Optimization Online, (under revision).

Given a graph $G = (V, E)$ with $|V| = n$ (vertices) and $|E| = m$ (edges)



Clique → a subset $K \subseteq V$ of vertices inducing a complete graph $G[K]$.

Maximum Clique: $K = \{v_1, v_2, v_3, v_4, v_5\}$ → clique number $\omega(G) = 5$

Research questions and motivation

- We are looking for the most vital (also called most vulnerable or most critical) vertices of a graph
- We are concerned in preserving (or limiting) the cohesiveness property.
- "tightly knit" and cohesive subgraphs are frequently identified using the notion of clique, i.e., a subset of vertices that are pairwise connected.

We study the problem of identifying a most vital subset of vertices with respect to the clique number.

- In the context of graph theory, we want to analyze the resilience of networks with respect to clique-interdiction attacks $\rightarrow$ removal of vertices

Clique-Interdiction curve of a graph

- the decrease of the size of the maximum clique as a function of an incremental interdiction budget

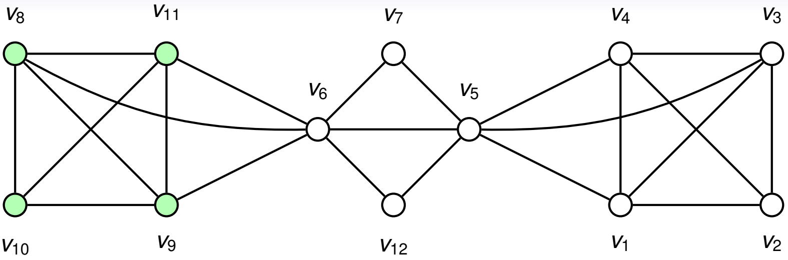
The Maximum Clique Vertex-Interdiction Problem (CIP)

- We study the two player zero-sum Stackelberg game in which the leader interdicts (removes) a maximum number of vertices from a simple graph, and the follower searches for the maximum clique in the interdicted graph.
- The goal of the leader is to derive an interdiction strategy which will result in the worst possible outcome for the follower.

Definition

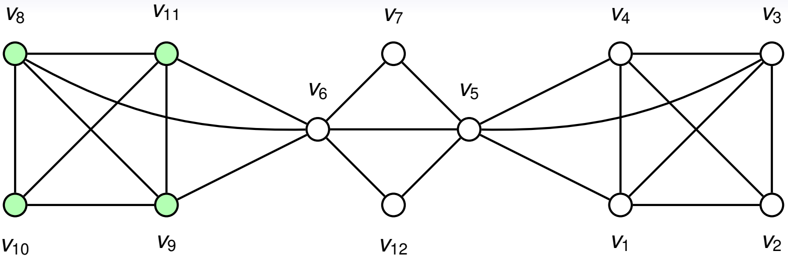
Given a graph G and an interdiction budget $k \in \mathbb{N}$, the maximum clique interdiction problem is to find a subset of at most k vertices to delete from G so that the size of the maximum clique in the remaining graph is minimized.

The set of interdicted vertices is called an interdiction strategy



The clique number is $\omega(G) = 4$ $\quad (K_2 = \{v_8, v_9, v_{10}, v_{11}\}$, there are others!)

An optimal interdiction strategy with $k = 2$ $\quad (\omega(G[V \setminus \{v_4, v_{11}\}]) = 3)$



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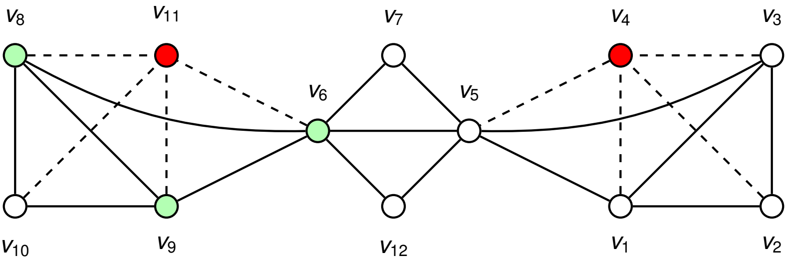

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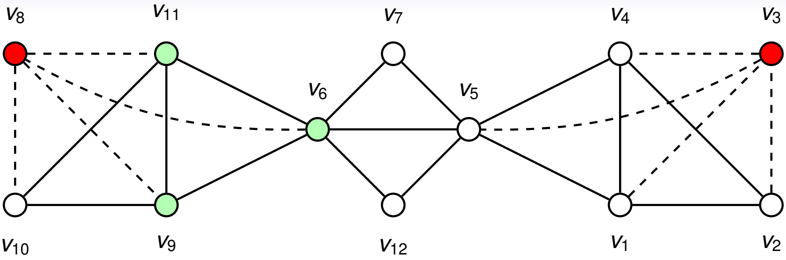
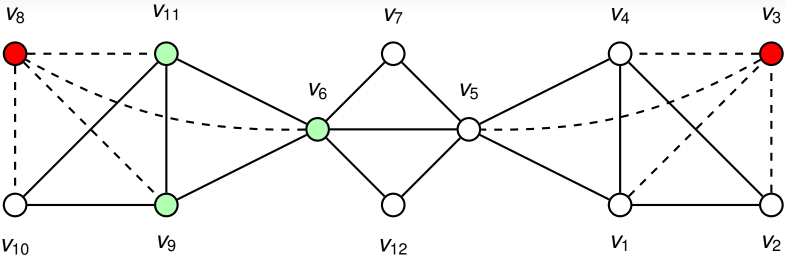


Another optimal interdiction strategy with $k = 2$ $(\omega(G[V \setminus \{v_3, v_8\}]) = 3)$

Another optimal interdiction strategy with $k = 2$ $(\omega(G[V \setminus \{v_4, v_9\}]) = 3)$



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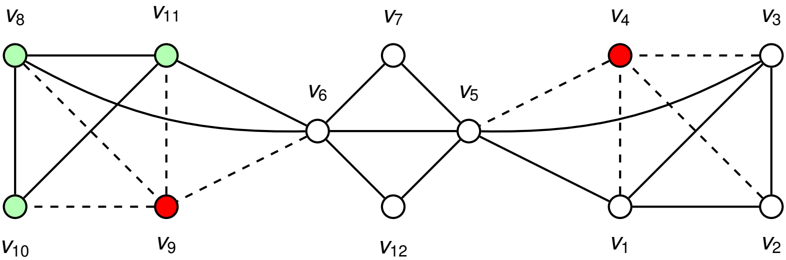
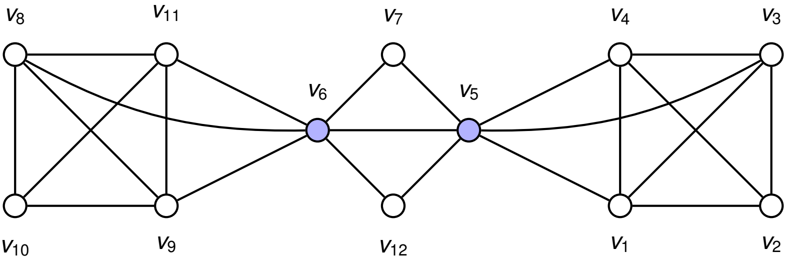


Another optimal interdiction strategy with $k = 2$ $(\omega(G[V \setminus \{v_4, v_9\}]) = 3)$

Centrality measure vs most vital nodes with respect to $\omega(G)$

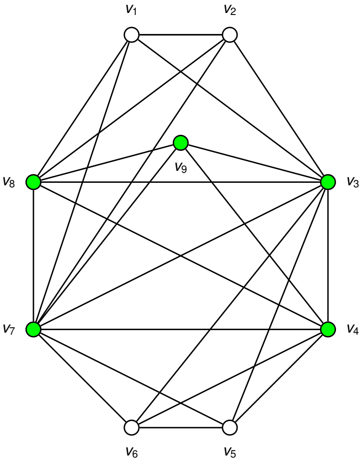


Other centrality measures rank the vertices $\{v_5, v_6\}$ as the most central ones.

- degree centrality: number of incident edges;
- closeness centrality: average length of the shortest path between the node and all other nodes in the graph.
- betweenness centrality: number of times a node acts as a bridge along the shortest path between two other nodes

$\{v_5, v_6\}$ are not the most critical ones for cohesiveness!

Example: $\omega(G) = 5$



Maximum Clique $\tilde{K} = \{v_3, v_4, v_7, v_8, v_9\}$

What is an optimal interdiction policy with $k = 2$?

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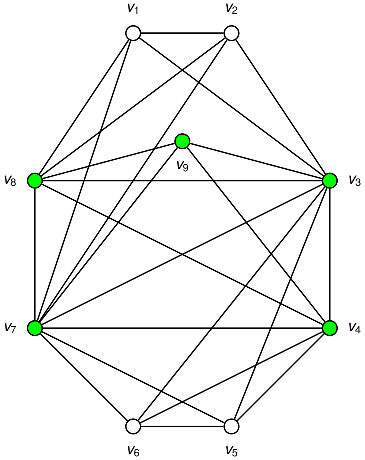
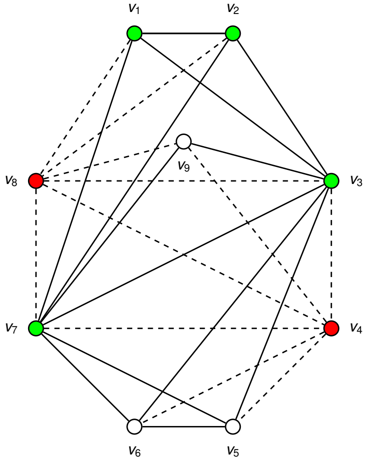




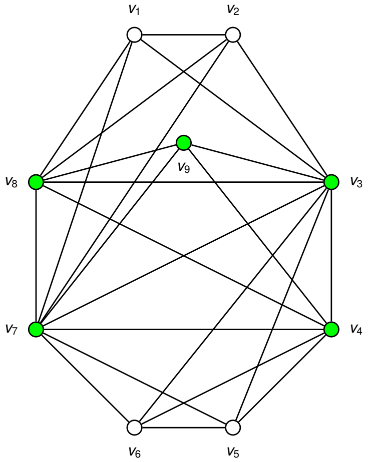
Maximum Clique $\tilde{K} = \{v_3, v_4, v_7, v_8, v_9\}$

An Optimal interdiction policy, $k = 2$

$$\omega(G[V \setminus \{v_4, v_8\}]) = 4$$

There are 2 cliques of size 4!

Example: $\omega(G) = 5$



What is an optimal interdiction policy with $k = 3$?

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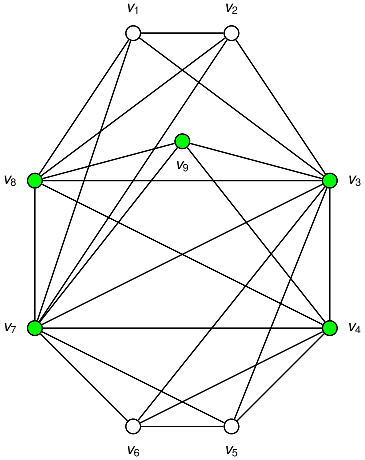
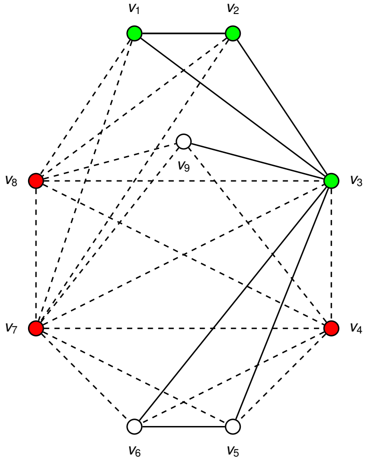
Example: $\omega(G) = 5$



Maximum Clique $\tilde{K} = \{v_3, v_4, v_7, v_8, v_9\}$

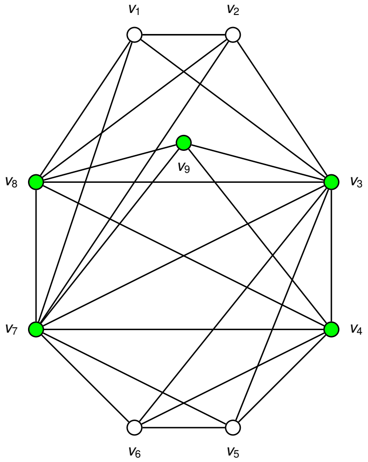


An Optimal interdiction policy, $k = 3$

$$\omega(G[V \setminus \{v_4, v_7, v_8\}]) = 3$$

There are 2 cliques of size 3!

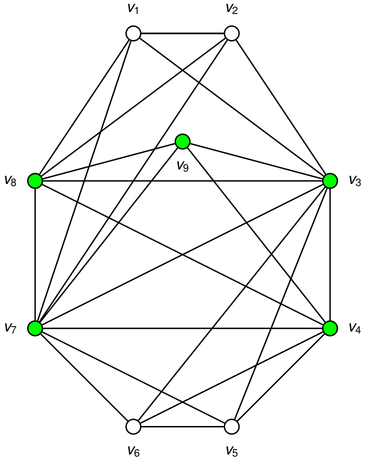
Example: $\omega(G) = 5$



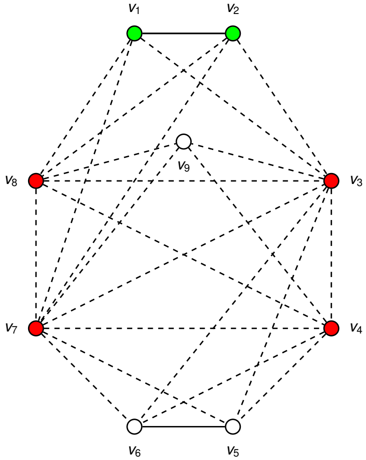
Maximum Clique $\tilde{K} = \{v_3, v_4, v_7, v_8, v_9\}$

What is an optimal interdiction policy with $k = 4$?

Example: $\omega(G) = 5$



Maximum Clique $\tilde{K} = \{v_3, v_4, v_7, v_8, v_9\}$



An Optimal interdiction policy, $k = 4$

$$\omega(G[V \setminus \{v_3, v_4, v_7, v_8\}]) = 2$$

There are 2 cliques of size 2!

Literature Overview

- No exact specialized algorithms for CIP exit in the literature
- CIP belongs to a larger family of Interdiction Games under Monotonicity (Fischetti et al. 16; focus on knapsack interdiction games).
- Games where the follower subproblem satisfies a monotonicity (or hereditary) property, exploited to derive a single-level integer linear programming formulation.

Related problems

- Minimum Vertex Blocker Clique Problem (Mahdavi Pajouh et al. 16), they tackle graphs with at most 200 vertices and most of the instances are unsolved
- Edge Interdiction Clique Game (Tang et al. 16), they tackle graphs with 15 vertices and most of the instances are unsolved

Complexity

Decision Version of CIP (d-CIP): Is there an interdiction strategy of size at most k such that the maximum clique in the interdicted graph is not greater than some given bound ℓ?

- Observe that the answer to the decision problem is YES if only if the optimal CIP solution is $\leq \ell - 1$.
- d-CIP is not in NP, to test whether the resulting graph does not contain a clique of size ℓ requires answering the decision version of:
  - the maximum clique problem (NP-complete).
- d-CIG has been also called **Generalized Node Deletion (GND)** problem

Proposition (Rutenburg1991,Rutenburg1994)

The decision version of CIP is Σ_2^P-complete.

Bi-Level ILP Formulation

$$w_u = \begin{cases} 1, & \text{if vertex } u \text{ is interdicted by the leader,} \\ 0, & \text{otherwise} \end{cases} \qquad u \in V$$

$$x_u = \begin{cases} 1, & \text{if vertex } u \text{ is used in the maximum clique of the follower,} \\ 0, & \text{otherwise} \end{cases} \qquad u \in V$$

- Let $\mathcal{W}$ be the set of incidence vectors of all feasible interdiction policies:

$$\mathcal{W} = \left\{ w \in \{0,1\}^n : \sum_{u \in V} w_u \leq k \right\}$$

- Let $\mathcal{K}$ be the set of incidence vectors of all cliques in the graph G:

$$\mathcal{K} = \left\{ x \in \{0,1\}^n : x_u + x_v \leq 1, uv \in \overline{E} \right\}$$

Property

CIG can be restated as follows:

$$\min_{w \in \mathcal{W}} \max_{K \in \mathcal{K}} \left\{ |K| - \sum_{u \in K} w_u \right\}. \qquad (0.1)$$

Bi-Level ILP Formulation

A new continuous variable ϑ → the size of max clique in the interdicted graph

$$\min \quad \vartheta \tag{0.2a}$$

$$\text{subject to} \quad \sum_{u \in V} w_u \leq k \tag{0.2b}$$

$$w_u \in \{0,1\} \qquad u \in V \tag{0.2c}$$

$$\text{where} \quad \vartheta = \max \sum_{u \in V} x_u \tag{0.2d}$$

$$\text{s.t.} \quad x_u \leq 1 - w_u \qquad u \in V \tag{0.2e}$$

$$x_u + x_v \leq 1 \qquad uv \in \overline{E} \tag{0.2f}$$

$$x_v \in \{0,1\} \qquad v \in V \tag{0.2g}$$

This formulation can be solved via a generic Solver for Mixed-Integer Bilevel Linear Problems, e.g.,
https://msinnl.github.io/pages/bilevel.html

Single-Level ILP Reformulation

For every feasible interdiction policy $\bar{w} \in \mathcal{W}$, the follower's problem becomes:

$$\max_{x \in \mathcal{K}} \left\{ \sum_{u \in V} x_u \; : \; x_u \leq 1 - \bar{w}_u, \; u \in V \right\} = \max_{x \in \mathcal{K}} \sum_{u \in V} x_u (1 - \bar{w}_u)$$

- Constraints of the follower independent from leader actions.

Proposition

The following is a valid ILP formulation for CIP:

$$\min_{w \in \mathcal{W}} \quad \vartheta \tag{0.3}$$

$$\vartheta \geq \underbrace{|K| - \sum_{u \in K} w_u}_{\text{size of } K \text{ in the interdicted graph}} \qquad K \in \mathcal{K} \tag{0.4}$$

- This model can be effectively solved via Branch and Cut!

Separation of the Clique Interdiction (CI) Cuts (integer points)

- Given a feasible realization (interdiction policy) $\bar{w} \in \mathcal{W}$ and the current value $\bar{\vartheta}$, we need to answer to the following question:

Are all the CI Cuts satisfied?

$$\bar{\vartheta} \geq \underbrace{|K| - \sum_{u \in K} \bar{w}_u}_{\textit{size of } K \textit{ in the interdicted graph}} \qquad K \in \mathcal{K}$$

- Separation Problem (SP):

$$\max \left\{ \sum_{u \in V} (1 - \bar{w}_u) x_u : \quad x_u + x_v \leq 1, \quad uv \in \overline{E} \right\}$$

- The Maximum Clique Problem in the interdicted graph $G[V \setminus V_{\bar{w}}]$
- Let $\bar{K}$ be the maximum clique: if $|\bar{K}| > \bar{\vartheta}$ then a violated CI cut is found

Exact Solution Framework – CLIQUE-INTER

(i) Effective separation procedure of the *Clique Interdiction* (CI) cuts:

- Specialized combinatorial branch-and-bound algorithm (IMCQ) for solving the maximum clique problem once the nodes of an interdiction policy have been removed from the graph G.
- Make the separated cliques maximal

(ii) Tight CIP upper and lower bounds (ℓ_{min} and ℓ_{max}):

- To initialize the lower bound value of the variable θ we used the global lower bound ℓ_{min} using node-disjoint maximum cliques
- To determine a high-quality feasible CIP solution of value ℓ_{max}, we apply a battery of effective sequential greedy heuristics.

(iii) The graph *Reduction Technique*:

- For large-scale real-world graphs the ILP formulation unless the input graph can be safely reduced to a smaller one.

Separating the Clique Interdiction Cuts with IMCQ

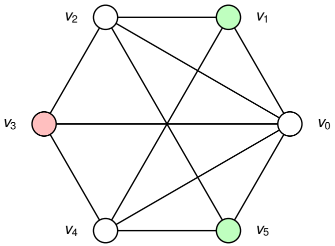
The separation problem requires solving the MCP in a number of induced subgraphs $G[V \setminus V_{\bar{w}}]$, where $V_{\bar{w}}$ is a feasible interdiction policy

- We have designed a combinatorial branch-and-bound (B&B) algorithm inspired by the ideas described in (Li 17) and (San Segundo16).
- Specialized *n*-ary branching scheme, based on the concept of Pruned and Branching Sets
- Using tight upper bounds on the infrachromatic bounding functions (potentially stronger than the chromatic number!)
- Plus! Compact bitstring representation both for vertex sets and the adjacency matrix and peeling procedures

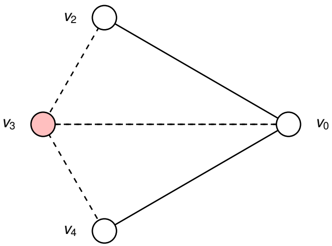
The Pruned and Branching Sets (Main Ideas)

- At each node of the branching tree we have: (*i*) a (non-maximal) clique $\hat{K} \subseteq V$ (feasible solution), (*ii*) a subproblem graph $\hat{G} = (\hat{V}, \hat{E})$ and (*iii*) a global lower bound $\tilde{\omega}$
- The subproblem graph is the Intersection of the neighboorhoods of the vertices in $\hat{K}$

$$\hat{V} = \bigcap_{v \in \hat{K}} N(v), \quad \text{and} \quad \hat{E} = E(\hat{V}).$$



Branching on $v_3 \rightarrow \hat{K} = \{v_3\}$



The subproblem graph $\hat{G}$

The Pruned and Branching Sets (Main Ideas)

- For a pair $(\hat{G}, \tilde{\omega})$ we want to determine a set $P \subseteq \hat{V}$ for which a MCP upper bound $UB(P)$ does not allows to improve the incumbent solution value $\tilde{\omega}$
- The Pruned and Branching Sets can be defined as follows:

$$P = \arg\max_{\hat{P} \subseteq \hat{V}} \left\{ |\hat{P}| : \tilde{\omega} - |\hat{K}| \geq UB(\hat{P}) \geq \omega(\hat{G}[\hat{P}]) \right\}, \quad \text{and} \quad B = \hat{V} \setminus P.$$

- By construction, P is the largest subset of $\hat{V}$ with the property that by branching on the vertices of P we cannot improve the incumbent solution value, since

$$\underbrace{\tilde{\omega}}_{Lower \quad Bound} \geq \underbrace{UB(P) + |\hat{K}|}_{Upper \quad Bound}$$

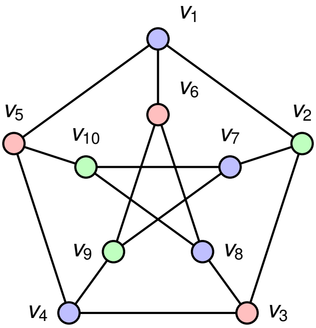
- For this reason, to improve $\tilde{\omega}$, one has to branch first on at least one vertex from the branching set B.

The Vertex Coloring Problem (VCP)

Given a graph $G = (V, E)$, the VCP asks for a partition of the vertex set

$$C = \{S_1, S_2, \ldots, S_k\},$$

with the min number of colors, s. t. vertices linked by an edge have diff colors.



$$S_1 = \{v_1, v_4, v_7, v_8\}$$

$$S_2 = \{v_2, v_9, v_{10}\}$$

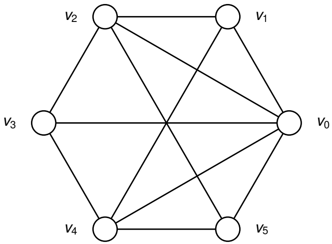
$$S_3 = \{v_3, v_6, v_5\}$$

chromatic number $\rightarrow \chi(G) = 3$

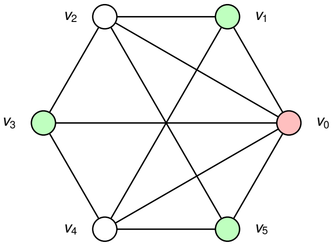
- A coloration C is a partition a of vertices into stables sets of G
- Clique number $\rightarrow \omega(G) = 2 \leq \chi(G)$

The Pruned and Branching Sets (Main Ideas)

- Computing the largest pruned set P is computationally challenging and in some case it is not useful
- For this reason we compute it heuristically using feasible coloring as upper bounds $UB(P)$
- Example: Consider the following subproblem graph, $\tilde{\omega} = 4$ and $|\hat{K}| = 2$



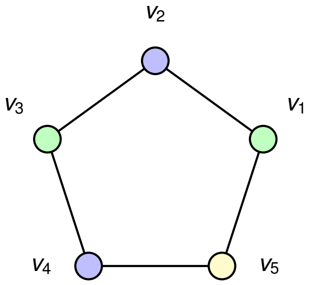
The subproblem graph $\hat{G}$



Pruned Set $B = \{v_0, v_1, v_3, v_5\}$
Branching Set $B = \{v_2, v_4\}$

Infra-chromatic Bounding Functions (Main Ideas)

Partial MAX-SAT Bound



cycle C of size 5
$\omega(C) = 2$, $\chi(C) = 3$

- Hard Clauses (non-edges)

$$h_1 \equiv \bar{x}_1 \vee \bar{x}_3, \; h_2 \equiv \bar{x}_1 \vee \bar{x}_4$$

$$h_3 \equiv \bar{x}_2 \vee \bar{x}_4, \; h_4 \equiv \bar{x}_2 \vee \bar{x}_5, \; h_5 \equiv \bar{x}_3 \vee \bar{x}_5$$

- Soft Clauses (colors)

$$s_1 \equiv x_1 \vee x_3, \; s_2 \equiv x_2 \vee x_4, \; s_3 \equiv x_5$$

- Unit Literal Propagation

$$x_5 = 1 \rightarrow x_2 = 0 \; (h_4) \rightarrow x_4 = 1 \; (s_2)$$

$$x_5 = 1 \rightarrow x_3 = 0 \; (h_5) \rightarrow x_1 = 1 \; (s_1)$$

- Inconsistency!
  $\rightarrow h_2 \quad \text{core} \quad \{s_1, s_2, s_3\}$
- Stronger Bound
  $\rightarrow \chi(C) > 3 - 1 = 2 \geq \omega(C)$

Computing the global lower bound $\ell_{\min}$

Proposition

Given a subgraph $G' = (V, E')$ with $E' \subset E$, the optimal CIP solution on G' provides a valid lower bound for the optimal CIP solution on G.

- rather counter-intuitive! reducing the input graph, instead of obtaining a valid upper bound for a minimization problem, we obtain a valid lower bound (the feasibility space of the follower is reduced)

Corollary

Given a set $\mathcal{Q}_{p+1} = (K_1, \dots, K_{p+1})$ of vertex-disjoint cliques of G, such that $|K_1| \geq \dots \geq |K_{p+1}|$, a valid lower bound $\ell_{\min}$ for the CIP can be obtained by computing

$$\ell_{\min} = \begin{cases} \max\left\{|K_{p+1}|, |K_p| - 1 - \left\lfloor \frac{k - k(\mathcal{Q}_p)}{p} \right\rfloor\right\}, & \text{if } k < k(\mathcal{Q}_{p^*+1}) \\ |K_{p+1}| - 1 - \left\lfloor \frac{k - k(\mathcal{Q}_{p+1})}{p+1} \right\rfloor, & \text{otherwise} \end{cases} \tag{0.5}$$

Where $k(\mathcal{Q}_q)$ denote the size of an optimal interdiction policy necessary to reduce the size of all cliques in $\mathcal{Q}_q$ to $|K_q| - 1$.

$$k(\mathcal{Q}_q) = q + \sum_{i=1}^{q-1} i \cdot (|K_i| - |K_{i+1}|).$$

Reducing the input graph

- The clique number of v is the size of the largest clique with v ($\omega_G(v)$) .
- The κ-core of a graph G is a maximal subgraph in which all vertices have degree at least κ
- The coreness-number of a vertex v, is equal to κ if v belongs to a κ-core but not to any $(\kappa + 1)$-core.

$$\omega_G(v) \leq \text{coreness}(v) + 1 \leq |N(v)| + 1 \qquad v \in V. \tag{0.6}$$

The following result identifies redundant vertices in the input graph G

Proposition

Let v be an arbitrary vertex from V. If $\omega_G(v) \leq \ell_{\text{opt}}$, then v cannot be part of a minimal optimal interdiction policy.

- instead of the (unknown) value of ℓ_{opt}, a lower bound $\ell_{\min}$
- instead of $\omega_G(v)$ (NP-hard), an upper bound $\text{coreness}(v) + 1$ (polynomial)

A MIP based upper bound

- The CIP can be formulated as follows:

$$\min_{w \in \mathcal{W}} \max \sum_{u \in V} (1 - w_u) x_u \tag{0.7a}$$

$$\text{s.t.} \quad x_u + x_v \leq 1 \qquad uv \in \overline{E} \tag{0.7b}$$

$$x_u \in \{0, 1\} \qquad u \in V \tag{0.7c}$$

- By relaxing the integrality constraints to $x_u \leq 1$, the dual of the inner maximization problem is:

$$\min_{(\alpha,\beta) \geq 0} \sum_{uv \in \overline{E}} \alpha_{uv} + \sum_{u \in V} \beta_u \tag{0.8a}$$

$$\text{s.t.} \quad \sum_{v \in \overline{\delta}(u)} \alpha_{uv} + \beta_u \geq 1 - w_u \qquad u \in V \tag{0.8b}$$

A MIP based upper bound

- By embedding this dual, we finally obtain a compact ILP single level model which we call U-CIP

$$
\text{(U-CIP)} \quad \min_{(\alpha,\beta)\geq 0} \sum_{uv\in\overline{E}} \alpha_{uv} + \sum_{u\in V} \beta_u
$$

$$
\text{s.t.} \qquad \sum_{uv\in E} w_{uv} \leq k
$$

$$
\sum_{v\in\bar{\delta}(u)} \alpha_{uv} + \beta_u \geq 1 - w_u \qquad u \in V
$$

$$
w_{uv} \in \{0,1\} \qquad uv \in E.
$$

- The U-CIP has a polynomial number of constraints and variables. However, the solution value of U-CIP only provides an upper bound for the CIP.
- An addition upper bound is computed by solving the MCP on the interdicted graph using the interdiction policy $\tilde{w}$ (optimal U-CIP solution)

Test-bed Instances

- Set A – Random Erdős-Rényi random $G(n, p)$ – 220 instances:
  - $n = |V| \in \{50, 75, 100, 125, 150\}$
  - $p \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 0.98\}$
  - $k \in \{\lceil 0.05 \cdot |V| \rceil, \lceil 0.1 \cdot |V| \rceil, \lceil 0.2 \cdot |V| \rceil, \lceil 0.4 \cdot |V| \rceil\}$
- Set B – Synthetic graphs – 32 instances:
  - Instances with $|V| = 200$ from the 2nd DIMACS challenge on Maximum Clique, Graph Coloring, and Satisfiability;
  - $k \in \{20, 40\}$
- Set C – Real-world (sparse) networks – 60 instances.
  - instances with up to $\approx 100,000$ nodes and $\approx 3,200,000$ edges.
  - Social Networks, Interaction networks, Recommendation networks, Collaboration networks, Technological networks, Scientific computing networks
  - $k \in \{\lceil 0.005 \cdot |V| \rceil, \lceil 0.01 \cdot |V| \rceil$

Performance Profile – Set A

1. CLIQUE−INTER: this is the benchmark setting of our exact algorithm, fully exploiting all its components.

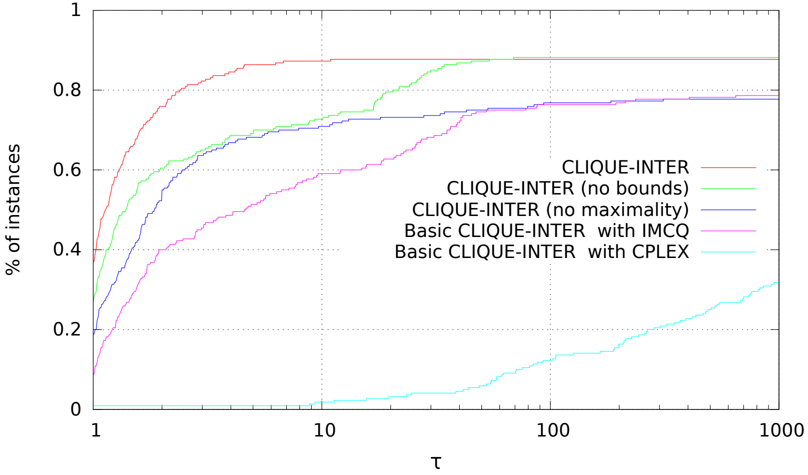
2. CLIQUE−INTER (no bounds): in this configuration we remove the use of CIP upper and lower bounds (ℓ_{min} and ℓ_{max}).

3. CLIQUE−INTER (no maximality): in this configuration we did not make maximal the cliques separated using IMCQ before adding the corresponding CIC.

4. Basic CLIQUE−INTER with IMCQ: in this configuration all components are removed, except the use of IMCQ to separate CICs.

5. Basic CLIQUE−INTER with CPLEX: this configuration corresponds to the basic branch-and-cut approach in which CICs are separated using CPLEX as a black-box clique solver applied to the classical clique ILP formulation.

Performance Profile – Set A



Comparison with state-of-the-art generic bilevel solver (BILEVEL)

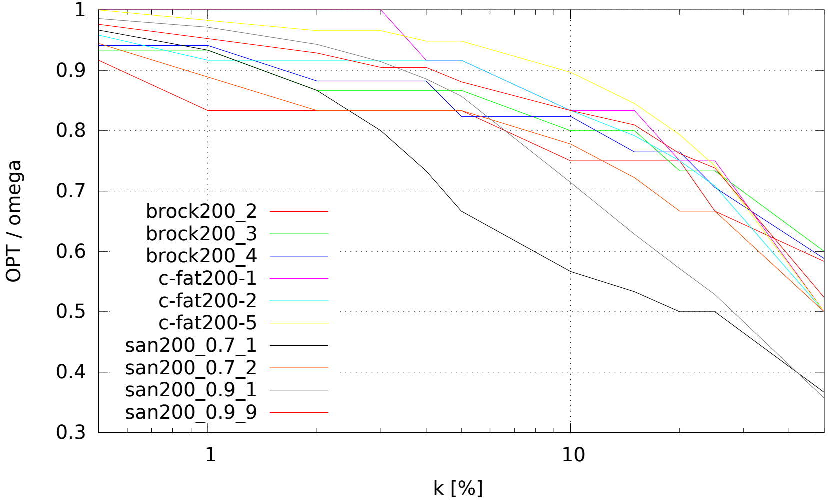
		CLIQUE-INTER				BILEVEL			
$\lvert V \rvert$	#	# solved	time	exit gap	root gap	# solved	time	exit gap	root gap
50	44	**44**	**0.01**	-	0.16	28	68.58	6.44	8.50
75	44	**44**	**1.45**	-	0.41	14	120.19	9.47	10.91
100	44	**37**	**9.30**	1.00	0.98	7	164.42	12.65	13.11
125	44	**35**	**13.43**	1.33	1.20	2	135.33	13.88	14.73
150	44	**33**	**27.23**	1.91	1.43	1	397.52	16.42	16.39

[1] Fischetti M, Ljubić I, Monaci M, Sinnl M.
A new general-purpose algorithm for mixed-integer bilevel linear programs.
Operations Research, 65(60):1615–1637, 2017.

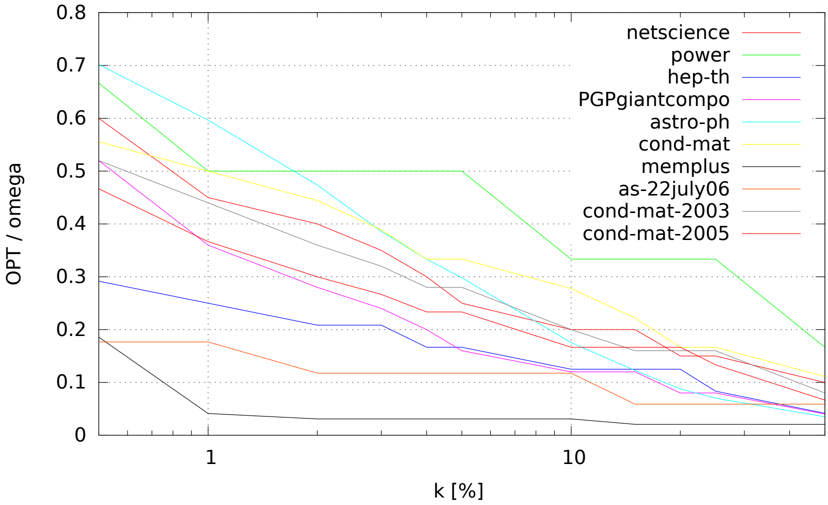
Results on Real-world (sparse) networks

	$\lvert V\rvert$	$\lvert E\rvert$	ω [s]	$k = \lceil 0.005 \cdot \lvert V\rvert \rceil$ [s]	$k = \lceil 0.005 \cdot \lvert V\rvert \rceil$ $\lvert V_p\rvert$	$k = \lceil 0.01 \cdot \lvert V\rvert \rceil$ [s]	$k = \lceil 0.01 \cdot \lvert V\rvert \rceil$ $\lvert V_p\rvert$
socfb-UIllinois	30,795	1,264,421	0.5	24.4	10,456	41.6	8290
ia-email-EU	32,430	54,397	0.0	0.6	30,375	0.5	29,212
rgg_n_2_15_s0	32,768	160,240	0.0	-	-	0.2	30,848
ia-enron-large	33,696	180,811	0.0	2.2	27,791	29.5	26,651
socfb-UF	35,111	1,465,654	0.3	17.8	14,264	87.8	10,708
socfb-Texas84	36,364	1,590,651	0.3	24.6	10,706	74.3	8,704
tech-internet-as	40,164	85,123	0.0	1.4	31,783	-	-
fe-body	45,087	163,734	0.1	1.8	2,259	1.8	2259
sc-nasasrb	54,870	1,311,227	0.1	-	-	145.5	1,195
soc-themarker_u	69,413	1,644,843	2.1	T.L.	35,678	T.L.	31,101
rec-eachmovie_u	74,424	1,634,743	0.7	-	-	367.3	13669
fe-tooth	78,136	452,591	0.5	18.9	7	19.0	7
sc-pkustk11	87,804	2,565,054	1.1	70.7	2,712	57.1	2,712
soc-BlogCatalog	88,784	2,093,195	11.7	T.L.	51,607	T.L.	46,240
ia-wiki-Talk	92,117	360,767	0.2	49.2	72,678	87.4	72,678
sc-pkustk13	94,893	3,260,967	1.3	724.9	2,360	879.2	2,354

Clique-Interdiction curve of a graph



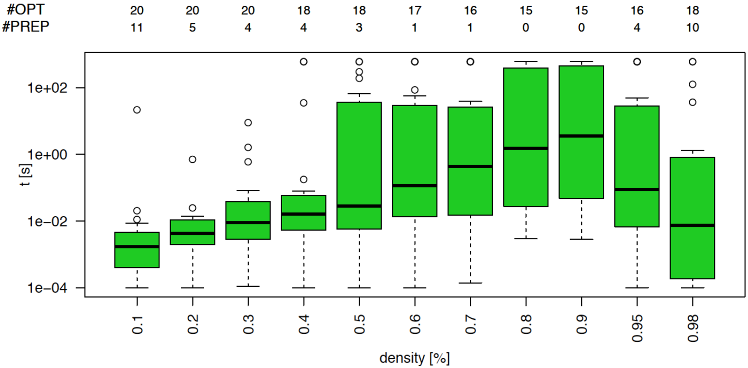
Clique-Interdiction curve of a graph



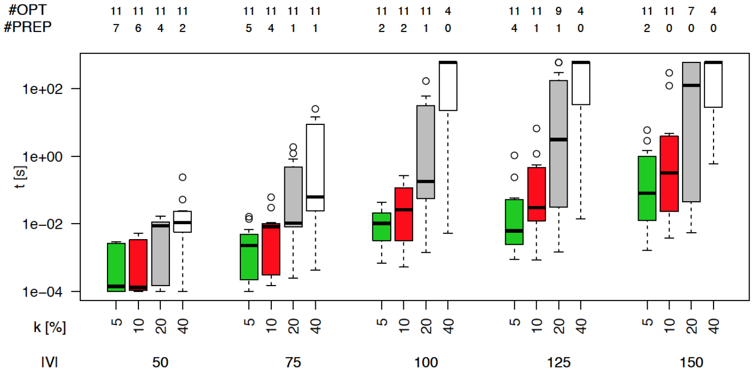
				CLIQUE-INTER $k = 20$					CLIQUE-INTER $k = 40$				
	μ	$\omega(G)$	time_ω	LB	UB	time	ℓ_{min}	ℓ_{max}	LB	UB	time	ℓ_{min}	ℓ_{max}
brock200_1	0.75	21	0.2	18	18	938.2	16	18	15	17	T.L.	13	17
brock200_2	0.50	12	0.0	9	9	0.1	8	10	8	9	T.L.	7	9
brock200_3	0.61	15	0.0	12	12	1.0	11	13	11	11	160.6	9	12
brock200_4	0.66	17	0.0	14	14	2421.8	12	15	12	13	T.L.	10	13
c-fat200-1	0.08	12	0.0	10	10	-	10	10	9	9	-	9	9
c-fat200-2	0.16	24	0.0	20	20	-	20	20	18	18	-	18	18
c-fat200-5	0.43	58	0.0	52	52	0.0	51	52	46	46	0.0	44	46
san200_0.7_1	0.70	30	0.0	17	17	5.4	16	18	15	15	134.4	14	17
san200_0.7_2	0.70	18	0.0	14	14	16.7	13	15	12	12	5.6	11	15
san200_0.9_1	0.90	70	0.0	50	50	-	50	50	40	40	13.3	39	49
san200_0.9_2	0.90	60	0.1	41	41	3.2	41	42	34	34	2266.9	33	41
san200_0.9_3	0.90	44	0.0	33	34	T.L.	32	37	28	31	T.L.	26	34
sanr200_0.7	0.70	18	0.1	15	15	29.2	14	16	13	14	T.L.	11	15
sanr200_0.9	0.90	42	1.9	33	35	T.L.	31	35	28	32	T.L.	25	33
gen200_p0.9_44	0.90	44	0.1	34	34	674.4	32	38	29	31	T.L.	26	36
gen200_p0.9_55	0.90	55	0.1	38	38	62.4	37	41	32	33	T.L.	29	40

Table 1: Computational results obtained by the CLIQUE-INTER on the instances with $|V| = 200$ from the 2nd DIMACS Challenge.

CPU times group by the graph density



CPU times group by the size and the interdiction budget



Facial study

Convex hull of feasible solutions of the CIP formulation

$$\mathcal{P}(G,k) = \operatorname{conv}\left\{ w \in \{0,1\}^{|V|}, \theta \geq 0 : \theta + \sum_{u \in K} w_u \geq |K|, \sum_{u \in V} w_u \leq k, K \in \mathcal{K} \right\}.$$

Proposition

The polytope $\mathcal{P}(G,k)$ is full dimensional.

Proposition

Let $u \in V$. The trivial inequality $w_u \leq 1$ defines a facet of $\mathcal{P}(G,k)$ if and only if $k \geq 2$.

Proposition

Let $u \in V$. The trivial inequality $w_u \geq 0$ defines a facet of $\mathcal{P}(G,k)$.

Lemma

Let $K \in \mathcal{K}$ be an arbitrary clique in G. If $|K| \leq \ell_{\text{opt}}$, then the associated clique interdiction inequality (0.4) *cannot define a facet.*

Facial study

Lemma

Let $K \in \mathcal{K}$ be an arbitrary clique in G. The inequality $\theta + \sum_{u \in K} w_u \geq |K|$ defines a facet only if K is maximal.

Lemma

Let K be a maximal clique and $v \in K$. If

$$\omega(G[V \setminus V']) \geq |K| - |V' \cap K| + 1 \quad \forall V' \subseteq V \text{ where } v \in V' \text{ and } |V'| \leq k, \quad (0.1)$$

then there exists $\alpha_v \leq 0$ such that the associated clique interdiction cut (0.4) can be down-lifted to

$$\theta + \sum_{u \in K \setminus \{v\}} w_u + \alpha_v w_v \geq |K|.$$

Corollary

Let $K \subset V$ be a clique. If there exists $v \in K$ satisfying (0.1) then the inequality (0.4) cannot define a facet.

Facial study

- the following Proposition provides necessary and sufficient conditions under which the CI cuts are facet defining.
- major theoretical result! it allows to characterize the strength of the ILP formulation upon which our solution framework is built on.

Theorem

Let $K \in \mathcal{K}$ be a maximal clique. Inequality (0.4) associated with K defines a facet of $\mathcal{P}(G, k)$ if and only if

- $|K| \geq \ell_{\text{opt}} + 1$
- *for all $v \in K$, there exists a subset $V' \subseteq V$ such that $v \in V'$, $|V'| \leq k$ and $\omega(G[V \setminus V']) + |V' \cap K| \leq |K|$.*

It is NP-hard to down-lift coefficients of a clique interdiction cut

- Heuristic lifting procedure! by underestimating the left-hand-side and overestimating the right-hand-side of the condition.

Conclusions

- We developed the first study on how to find the most vital k vertices of a graph, so as to reduce its clique number
- We derived tight combinatorial lower and upper bounds
- We derive a single-level reformulation based on an exponential family of Clique-Interdiction Cuts
- We provide necessary and sufficient conditions under which these cuts are facet defining and we propose a fast lifting procedures
- We developed a state-of-the-art algorithm for finding maximum cliques in interdicted graphs
- Social Networks are "vulnerable" to vertex-deletion attacks!

THANKS FOR YOUR ATTENTION!!!!

References

[1] F. F., I. Ljubić, P. San Segundo, and S. Martin.
The maximum clique interdiction problem.
European Journal of Operational Research, 277(1), 112-127, 2019.

[2] F. F., I. Ljubić, P. San Segundo, and Y. Zhao.
A branch-and-cut algorithm for the Edge Interdiction Clique Problem.
Optimization Online, (under revision).