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Polynomial optimization with the Lasserre hierarchy

5 March 2021

Outline

1. Polynomial optimization
2. Linear conic reformulation
3. Polynomial sums of squares and moments
4. The Lasserre hierarchy
5. Extensions

1 - Polynomial optimization

POP (Polynomial Optimization Problem)

Given polynomials $p, p_1, \ldots p_m \in \mathbb{R}[x]$ of the indeterminate $x \in \mathbb{R}^n$, consider the **nonlinear nonconvex** global optimization problem

$$p^* = \min_{x \in X} p(x)$$

defined on the bounded basic semialgebraic set

$$X := \{x \in \mathbb{R}^n : p_1(x) \geq 0, \ \ldots \ p_m(x) \geq 0\}$$

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In general POP can be very challenging:

- p can be nonconvex
- X can be nonconvex and/or disconnected and/or discrete
- there can be several global optimizers, maybe infinitly many

2 - Linear conic reformulation

Primal linear reformulation

Instead of the POP

$$p^* = \min_{x \in X} \; p(x)$$

over vectors in X, consider the **linear** problem (LP)

$$p^*_M = \min_{\mu} \; \int_X p(x) d\mu(x)$$

over probability measures (normalized bounded linear functionals on continuous functions) on X

> **Lemma:** $p^*_M = p^*$ and the LP has for optimal solution the Dirac measure at any optimal solution of the POP

Dual linear reformulation

The Lagrange dual to the LP on probability measures

$$\min_{\mu} \int_X p(x) d\mu(x)$$

reads

$$\max_{p_L} \; p_L \text{ s.t. } p(x) \geq p_L \; \forall x \in X$$

which can be rephrased as an LP on **positive polynomials**

$$\max_{p_L} \; p_L \text{ s.t. } p(x) - p_L \in P(X)_d$$

where $P(X)_d$ denotes the convex cone of polynomials of degree up to d that are non-negative on X

Moments

Let $(b_a(x))_{a\in\mathbb{N}^n_d}$ denote a basis of the vector space of n-variate polynomials of degree at most d of dimension $\binom{n+d}{n}$, indexed in $\mathbb{N}^n_d := \{a \in \mathbb{N}^n : \sum_{k=1}^n a_k \leq d\}$

The polynomial p can then be written as

$$p(x) = \sum_{a\in\mathbb{N}^n_d} p_a b_a(x)$$

and the objective function can be written as

$$\int_X p(x)d\mu(x) = \sum_{a\in\mathbb{N}^n_d} p_a y_a$$

which is a linear function of the **moments** of measure μ

$$y_a = \int_X b_a(x)d\mu(x)$$

Moments and positive polynomials

The LP on probability measures

$$\min_{\mu} \int_X p(x) d\mu(x)$$

becomes an LP on **moments**

$$\min_{y} \sum_a p_a y_a \text{ s.t. } y_0 = 1, \ y \in P(X)'_d$$

which is dual to the LP on **positive polynomials**

$$\max_{p_L} \ p_L \text{ s.t. } p(x) - p_L \in P(X)_d$$

since

• measures on compact X are uniquely determined by moments
• the constraint $y_0 = 1$ corresponds to the normalization
• by the Riesz-Haviland Theorem, the cone of moments is dual to the cone of positive polynomials

Wonderful but ...

Challenging convex cones

Testing whether $p \in P(X)_d$ or $y \in P(X)'_d$ is difficult

Not much is known about the geometry of these cones

No efficient barrier function is known

... so we will content ourselves with **approximations**

3 - Polynomial sums of squares and moments

Approximating positive polynomials

The cone of positive polynomials $P(X)_d$ on the compact set

$$X := \{x \in \mathbb{R}^n : p_k(x) \geq 0, k = 1, \ldots, m\}$$

is generally **intractable**, so we will approximate it.

Denoting $p_0(x) := 1$ and enforcing (without loss of generality) $p_1(x) := R^2 - \sum_{i=1}^n x_i^2$ for R large enough, consider for $r \geq d$

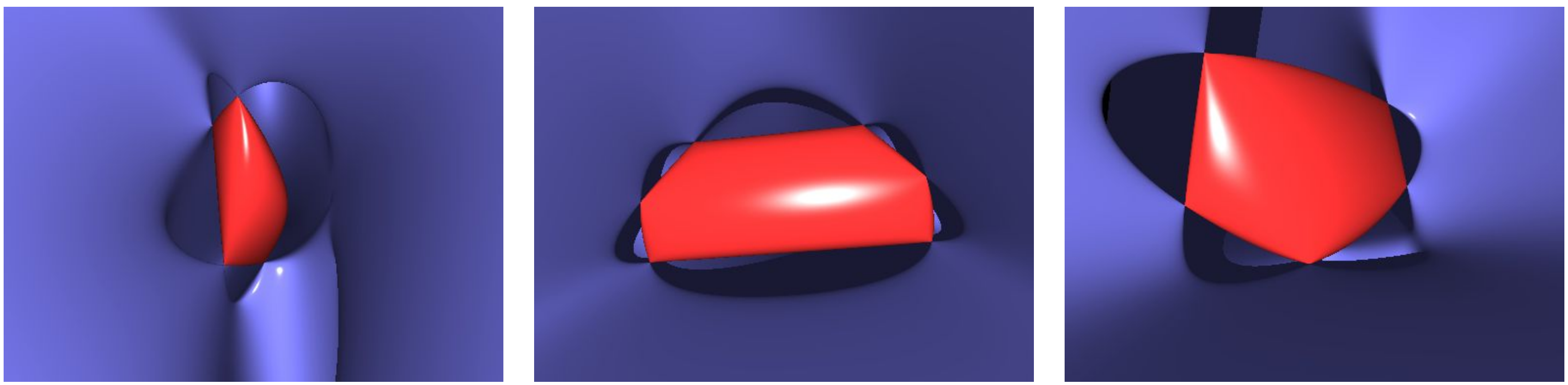
$$\Sigma(X)_r := \{p \in \mathbb{R}[x]_d : p = \sum_{k=0}^{m} s_k p_k, \;\; s_k \in \Sigma_{r-\deg p_k}\}$$

where Σ_d denotes the cone of polynomial **sums of squares** (SOS) of degree at most d

Lemma: By construction $\Sigma(X)_r \subset \Sigma(X)_{r+1} \subset P(X)_d$

Polynomial SOS

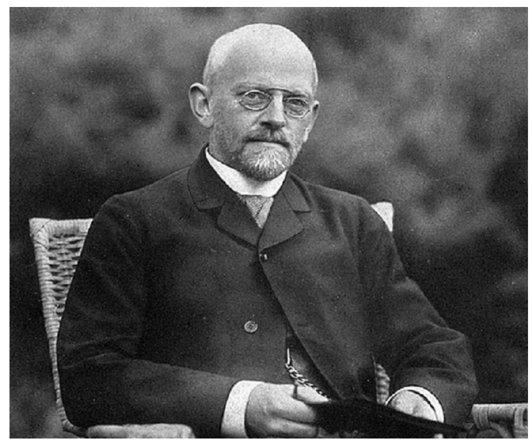
> **Lemma:** Deciding whether a polynomial is SOS reduces to semidefinite programming



Semidefinite programs can be solved efficiently with primal-dual interior-point methods

SOS and positivity

Theorem (Hilbert 1888): $\Sigma(\mathbb{R}^n)_{2d} = P(\mathbb{R}^n)_{2d}$ if and only if $n = 1$ or $d = 1$ or $n = d = 2$



Hilbert's 17th problem at ICM Paris 1900

Motzkin's 1965 example
$p = 1 - 3x_1^2x_2^2 + x_1^4x_2^2 + x_1^2x_2^4 \in P(\mathbb{R}^2)_6 \backslash \Sigma(\mathbb{R}^2)_6$

Moment relaxations

Hence we have a hierarchy of tractable **inner approximations** for the cone of positive polynomials

$$\Sigma(X)_r \subset \Sigma(X)_{r+1} \subset P(X)_d$$

Using convex duality, we also have a hierarchy of tractable **outer approximations** for the cone of moments

$$\Sigma(X)'_r \supset \Sigma(X)'_{r+1} \supset P(X)'_d$$

Elements of $\Sigma(X)'_r$ are sometimes called pseudo-expectations or pseudo-moments, since some of them are not moments

We also say that $\Sigma(X)'_r$ is a **relaxation** of $P(X)'_d$

4 - The Lasserre hierarchy

The Lasserre or moment-SOS hierarchy

Replace the intractable problems

$$p^* = \min_y \sum_a p_a y_a \text{ s.t. } y_0 = 1,\ y \in P(X)'_d$$

$$p^* = \max_{p_L}\ p_L \text{ s.t. } p(x) - p_L \in P(X)_d$$

with the hierarchy of **semidefinite** problems

$$p^*_r = \min_y \sum_a p_a y_a \text{ s.t. } y_0 = 1,\ y \in \Sigma(X)'_r$$

$$p^*_r = \max_{p_L}\ p_L \text{ s.t. } p(x) - p_L \in \Sigma(X)_r$$

for increasing values of $r \geq d$

Convergence

Integer r is called the **relaxation order**

Since $\Sigma(X)_r \subset \Sigma(X)_{r+1} \subset P(X)_d$, we have a monotone non-decreasing sequence of lower bounds on the POP value:

$$p_r^* \leq p_{r+1}^* \leq p^*$$

Theorem (Putinar 1993): $\overline{\Sigma(X)_\infty} = P(X)_d$

Theorem (Lasserre 2001): $p_\infty^* = p^*$

Finite convergence

Theorem (Nie 2014): Generically $\exists r < \infty$ such that $p_r^* = p^*$

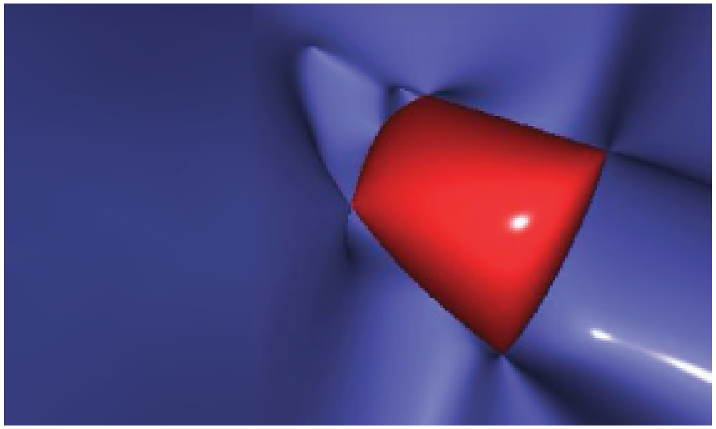
In other words, a vanishing small random perturbation of the input data of a given POP ensures **finite convergence** of the Lasserre hierarchy

We also have sufficient linear algebra conditions to ensure finite convergence, **certify** global optimality and **extract** minimizers

We can use the Christoffel-Darboux SOS polynomial to retrieve approximately the **variety** of global minimizers

Extensions

Dynamical systems: polynomial iterations
Maximal positively invariant sets
Optimal control of polynomial ODEs



POEMA European Network (2019-2022) poema-network.eu

Lectures at homepages.laas.fr/henrion/courses/poema20

MINOA

MIXED-INTEGER NON-LINEAR OPTIMISATION: ALGORITHMS AND APPLICATIONS

Didier Henrion

Solving numerically polynomial optimization problems

5 March 2021

One page summary of previous talk

The polynomial optimization problem

$$p^* = \min_{x \in X} p(x)$$

defined on the bounded basic semialgebraic set

$$X := \{x \in \mathbb{R}^n : p_1(x) \geq 0, \ldots p_m(x) \geq 0\}$$

can be solved by a hierarchy of semidefinite problems

$$p^*_r = \min_y \sum_a p_a y_a \text{ s.t. } y_0 = 1, \; y \in \Sigma(X)'_r \quad \textbf{(moments)}$$

$$p^*_r = \max_{p_L} \; p_L \text{ s.t. } p(x) - p_L \in \Sigma(X)_r \quad \textbf{(sums of squares)}$$

with convergence guarantee $p^*_r \leq p^*_{r+1} \leq p^*_\infty = p^*$

Demo topics

SOS reduces to SDP

Dual to SOS are moment matrix LMI relaxations

Low rank moment matrices for global optimality

Christoffel-Darboux polynomial for optimal variety

... and maybe more depending on your interest ?

We will be using `GloptiPoly` and `SeDuMi` on `Matlab`