Multiobjective Mixed Integer Nonlinear Programming (MOMINLP): decision and criterion space search algorithms

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Outline of the lecture

- Introduction to MOMINLP
  - formulation of the problem
  - basic definitions
  - solution approaches

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- `FPA`: a criterion space search algorithm for bi-objective integer nonlinear programming problems

- `MOMIX`: a decision space search algorithm for multi-objective mixed integer convex programming problems

Problem Formulation

Multiobjective Mixed Integer Nonlinear programming problems (MOMINLPs) can be formulated as follows:

$$\begin{aligned} \min \quad & (f_1(x), \ldots, f_m(x))^T \\ \text{s.t.} \quad & g_k(x) \leq 0 \quad k = 1, \ldots, p \\ & x_i \in \mathbb{Z} \quad \forall i \in I, \end{aligned} \tag{MOMINLP}$$

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where

- $f_j, g_k : \mathbb{R}^n \to \mathbb{R}; \; j = 1, \ldots, m; \; k = 1, \ldots, p$
- the index set $I \subseteq \{1, \ldots, n\}$ specifies which variables have to take integer values

Motivation

Multiobjective mixed integer optimization problems arise in **many application fields** such as

- engineering
- finance
- design of water distribution networks
- location or production planning
- emergency management

see e.g. [Pecci et al. OPTE (2018)], [Yenisey et al. Omega (2014)], [Liu et al. C&OR (2014)], [Xinodas et al. JOGO (2010)], [Ehrgott et al. INFOR (2009)]

Basic definitions

- point $x^* \in \mathcal{F}$ is **efficient** for (MOMIC) if there is no $x \in \mathcal{F}$ with $f(x) \leq f(x^*)$ and $f(x) \neq f(x^*)$

  The set of efficient points for (MOMIC) is the **efficient set** of (MOMIC)

- point $z^* = f(x^*) \in \mathbb{R}^m$ is **nondominated** for (MOMIC) if $x^* \in \mathcal{F}$ is an efficient point for (MOMIC)

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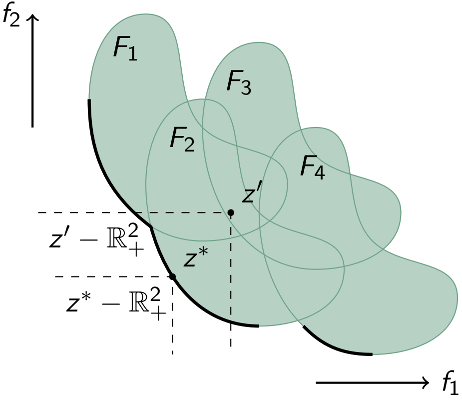
- Let $x^*, x \in \mathcal{F}$ with $f(x^*) \leq f(x)$ and $f(x^*) \neq f(x)$

  Then we say that x^* **dominates** x and also that $f(x^*)$ **dominates** $f(x)$

Challenges of multiobjective mixed integer programming

Example: image set of a bi-objective instance



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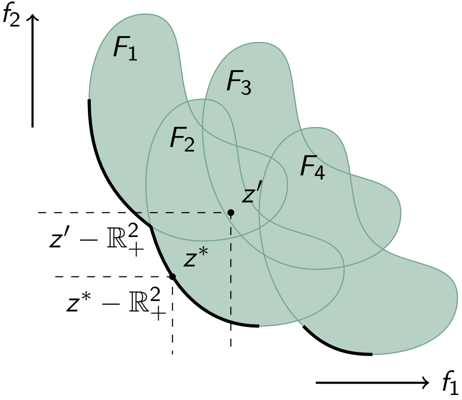

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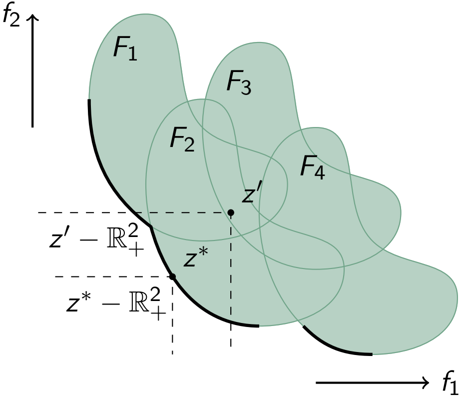

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- z^* is a nondominated point and the preimage of z^* is an efficient point

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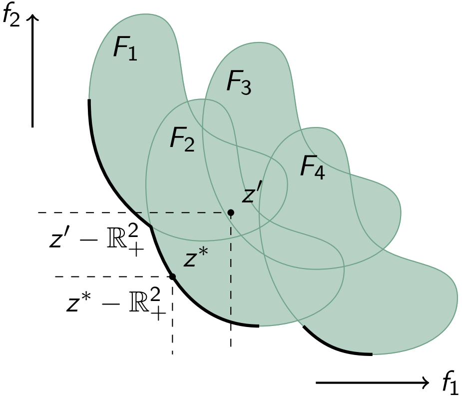

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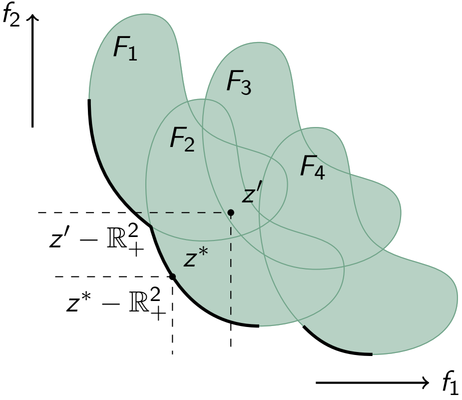

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- z^* is a nondominated point and the preimage of z^* is an efficient point
- z' is dominated because $z^* \leq z'$ and $z^* \neq z'$.
- all the points $z \in F_3$ are dominated

Solution approaches

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- **Criterion space search algorithms**:
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  find non-dominated points by addressing a **sequence of single-objective optimization problems**

- **Decision space search algorithms**:
  approaches that work in the space of decision variables

  extend approaches developed for single-objective MINLPs to the case of multiple objectives

FPA: a criterion space search algorithm for bi-objective integer nonlinear programming problems

M. De Santis, G. Grani, L. Palagi
Branching with hyperplanes in the criterion space: The frontier partitioner algorithm for bi-objective integer programming.
European Journal of Operational Research, 283(1), 57-69 (2020)

Problem Formulation

We address **bi-objective integer programming problems**

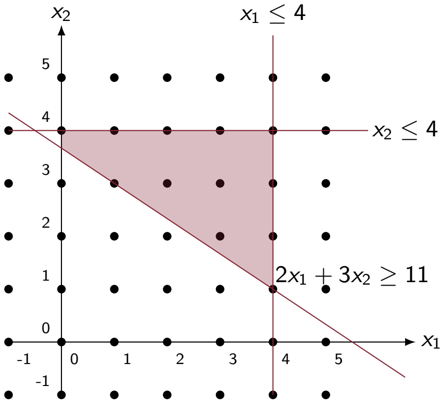
$$\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} (f_1(x), f_2(x)) \qquad \text{(BOIP)}$$

where

- $\mathcal{X} \subseteq \mathbb{R}^n$
- $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ are continuous

Example:

[Ehrgott "Multicriteria Optimization" - 2005]



$$
\begin{aligned}
\text{min} \quad & (x_1, x_2) \\
\text{s.t.} \quad & 2x_1 + 3x_2 \geq 11 \\
& x \in [0, 4] \cap \mathbb{Z}^2
\end{aligned}
$$

Example: $\mathcal{Y}_N = \{(0,4);\ (1,3);\ (3,2);\ (4,1)\}$

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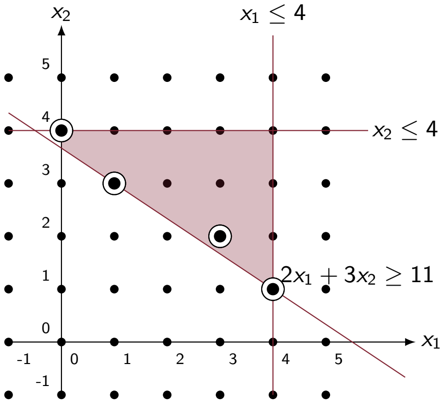

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Criterion space algorithms

Criterion space search algorithms find non-dominated points by addressing a **sequence of single-objective optimization problems**

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Criterion space search algorithms find non-dominated points by addressing a **sequence of single-objective optimization problems**

Once a non-dominated point is computed, the **dominated parts of the criterion space are removed** and the algorithms go on looking for **new** non-dominated points

Criterion space algorithms

solving single-objective optimization problems to get non-dominated points

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solving single-objective optimization problems to get non-dominated points

We refer to the **weighted-sum scalarization problem (INLP)** defined as

$$\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} \lambda_1 f_1(x) + \lambda_2 f_2(x) \qquad \text{(INLP)}$$

where $\lambda_1 + \lambda_2 = 1$, with $\lambda_i \geq 0, \;$ for $i = 1, 2$

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Proposition

Let $\lambda_1, \lambda_2 > 0$, then each solution of Problem (INLP) *is an efficient solution for Problem* (BOIP)

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The converse is true **only under proper convexity assumptions!!**

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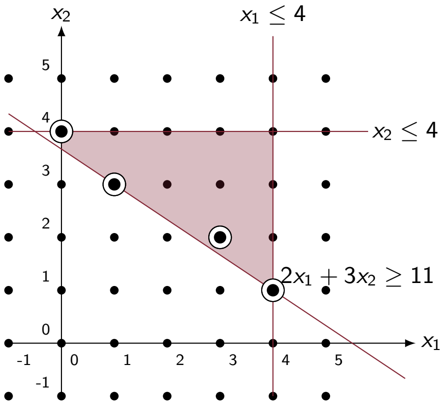

Point $(3,2)^\top$ **cannot be found** by weighted-sum scalarization!!

The Frontier Partitioner Algorithm FPA

Key ingredients

FPA is a **Criterion Space search Algorithm**

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At each iteration FPA:

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FPA is a **Criterion Space search Algorithm**

At each iteration FPA:

- computes one non-dominated solution (when it exists) **addressing a weighted-sum scalarization problem**
- in case a non-dominated solution is found, **two subproblems are constructed** using properly defined inequalities

The Frontier Partitioner Algorithm FPA

Positive gap assumption

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$. We say that f is a positive γ-function if $\gamma \in \mathbb{R}_+$ exists such that $|f(x) - f(z)| \geq \gamma$, for all $x, z \in \mathcal{X} \cap \mathbb{Z}^n$ with $f(x) \neq f(z)$.

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Assumption (Positive gap value)

The functions $f_i : \mathbb{R}^n \to \mathbb{R}$ in (BOIP) are positive γ-functions

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Definition of the inequalities

Let $\hat{y}^k$ be a **non-dominated point** for (BOIP) found at iteration k

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Remark

The inequalities $f_i(x) \leq \hat{y}_i^k - \epsilon_i, \; i = 1, 2$
cut the non-dominated solution $\hat{y}^k$
and they are **linear in the criterion space**

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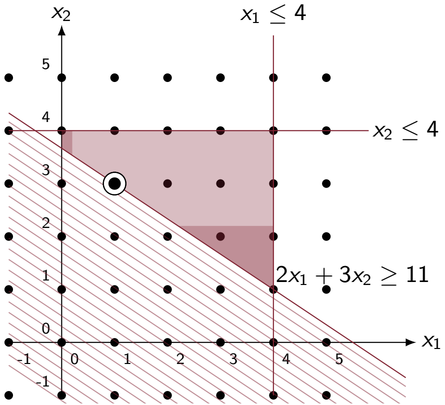
Definition of the inequalities



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Starting from $\hat{y}^0$, FPA **defines the following two BOIPs**:

$$\min_{x \in \mathcal{X}_1 \cap \mathbb{Z}^n} (f_1(x), f_2(x)) \qquad \mathcal{X}_1 = \mathcal{X} \cap \{x \in \mathbb{R}^n \,:\, f_1(x) \leq \hat{y}_1^0 - \epsilon_1\}$$

$$\min_{x \in \mathcal{X}_2 \cap \mathbb{Z}^n} (f_1(x), f_2(x)) \qquad \mathcal{X}_2 = \mathcal{X} \cap \{x \in \mathbb{R}^n \,:\, f_2(x) \leq \hat{y}_2^0 - \epsilon_2\}$$

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...and goes on **producing iteratively a finite lists of BOIPs!**

The Frontier Partitioner Algorithm FPA

Convergence analysis

Proposition

At every iteration FPA either states that the **BOIP** *considered* **is infeasible** *or finds a* **yet unknown non-dominated solution***.*

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Theorem

The Frontier Partitioner Algorithm **returns the complete Pareto frontier** $\mathcal{Y}_N$ *of (BOIP) after having addressed* $2|\mathcal{Y}_N| + 1$ *single-objective integer programs.*

Improving the complexity of FPA

Use smart weights

In order to identify all $|\mathcal{Y}_N|$ non-dominated points of a BOIP by solving a sequence of subproblems, **any criterion space algorithm for BOIPs must solve at least $|\mathcal{Y}_N|$ subproblems**

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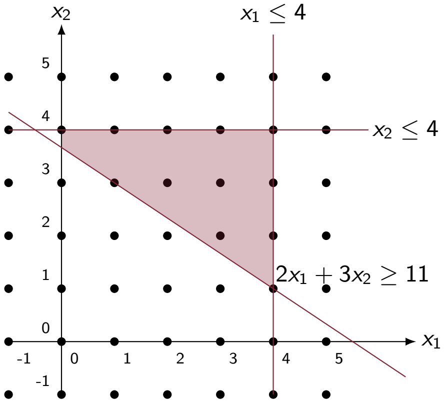
Remark

We can **drop down the complexity of** *FPA*
from $2|\mathcal{Y}_N| + 1$ *to* $|\mathcal{Y}_N| + 1$ *using smart weights!*

FPA applied to the example

smart weights



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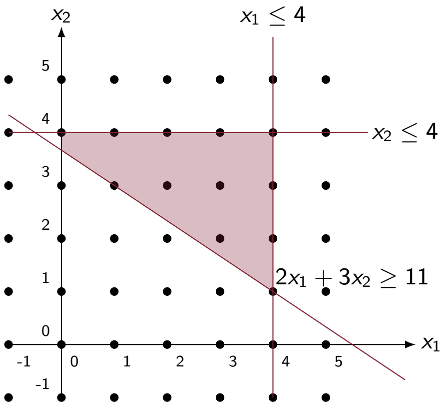

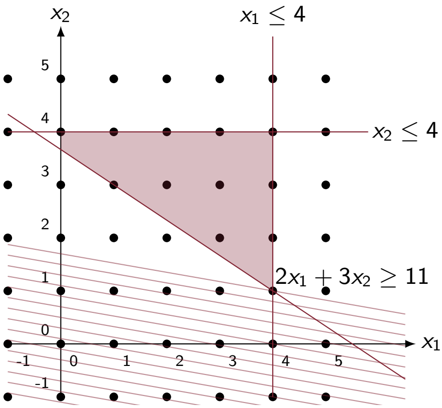

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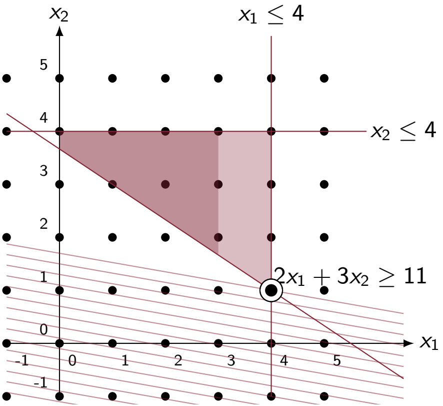

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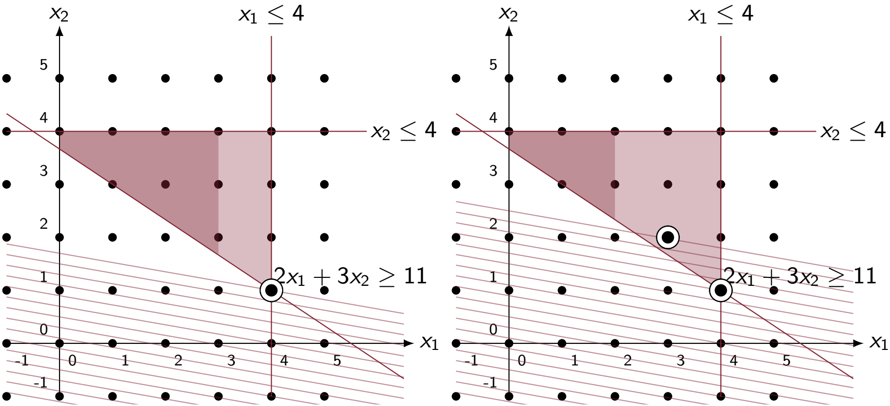

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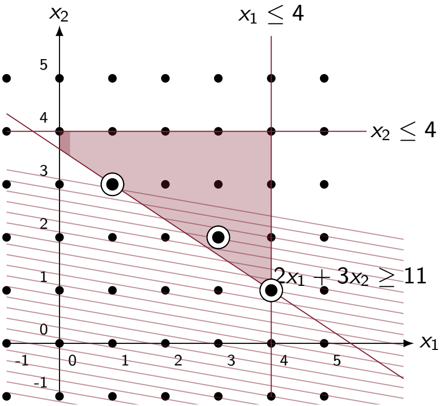

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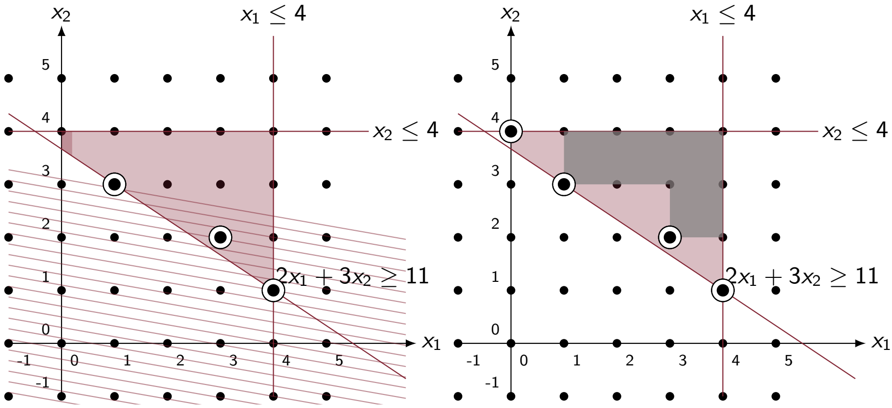

Which BOIPs can be addressed by FPA?

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$f_i(x) =$	γ	INLP oracle
$c^\intercal x$ with $c \in \mathbb{Z}^n$	1	*ILP*
$c^\intercal x$ with $c \in \mathbb{Q}^n$	$\frac{1}{r}$	*ILP*
$x^\intercal Qx + c^\intercal x$ with $Q \succeq 0,\ Q \in \mathbb{Z}^{n\times n},\ c \in \mathbb{Z}^n$	1	*QCQIP*
$x^\intercal Qx + c^\intercal x$ with $Q \succeq 0,\ Q \in \mathbb{Q}^{n\times n},\ c \in \mathbb{Q}^n$	$\frac{1}{r}$	*QCQIP*
$: \mathbb{Z}^n \to \mathbb{Z}$, convex	1	*CIP*

Table: Classes of functions that satisfy the positive gap value assumption.

Numerical results

Algorithm `FPA`

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- uses `CPLEX 12.7.1` to address the scalarized problem (INLP)

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- biobjective integer convex quadratic instances

Comparison via performance profiles

[Dolan, E. and Moré, J. (2002). *Benchmarking optimization software with performance profiles.* Mathematical Programming, 91, 201–213.]

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We define the **performance ratio**

$$r_{p,s} = t_{p,s} / \min\{t_{p,s'} : s' \in \mathcal{S}\},$$

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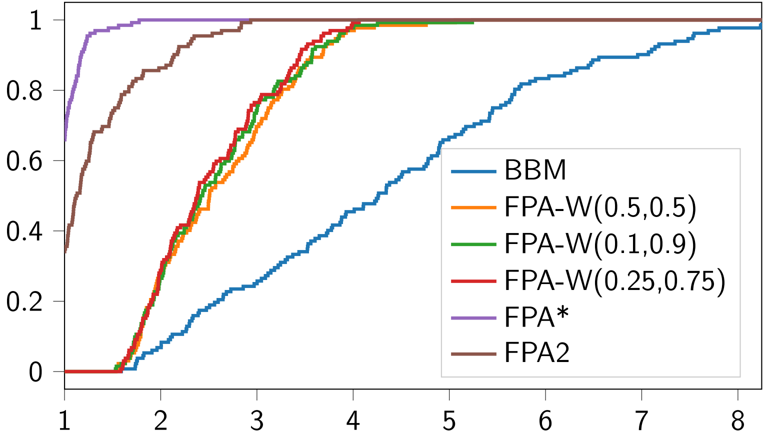
The **performance profile** for $s \in S$ is the plot of the **cumulative distribution function** ρ_s:

$$\rho_s(\tau) = |\{p \in \mathcal{P} : \; r_{p,s} \leq \tau\}| / |\mathcal{P}|$$

Results on biobjective integer linear instances

Performance profiles related to the CPU time



Results on biobjective integer quadratic instances

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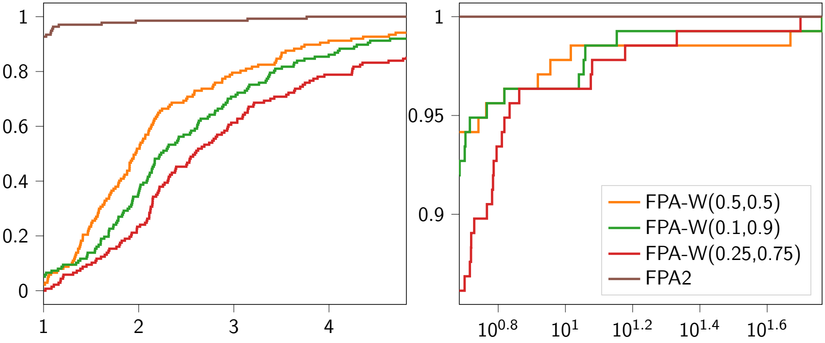

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FPA is a **criterion space algorithm** for **biobjective integer programming problems** that

- Can handle several classes of biobjective integer **nonlinear** programming problems
- It is based on the use of **properly defined inequalities**
- Has the complexity of $|\mathcal{Y}_N| + 1$
- On biobjective integer linear programming problems outperforms existing state-of-the art methods

MOMIX: a decision space search method for multi-objective mixed integer convex programming problems

M. De Santis, G. Eichfelder, J. Niebling, S. Rocktäschel
Solving multiobjective mixed integer convex optimization problems,
SIAM Journal on Optimization 30 (4), 3122-3145 (2020)

MOMIX: a decision space search method for (MOMIC)

MOMIX adresses **Multiobjective Mixed Integer Nonlinear** programming problems of the following form:

$$\begin{aligned} \min \quad & (f_1(x), \dots, f_m(x))^T \\ \text{s.t.} \quad & g_k(x) \le 0 \quad k = 1, \dots, p \\ & x \in B := [l, u] \\ & x_i \in \mathbb{Z} \quad \forall i \in I, \end{aligned} \tag{MOMIC}$$

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where

- $f_j, g_k : B \to \mathbb{R};\ j = 1, \ldots, m;\ k = 1, \ldots, p$ **convex and differentiable**
- $l, u \in \mathbb{R}^n$ are lower and upper bounds on the decision variables
- the index set $I \subseteq \{1, \ldots, n\}$ specifies which variables have to take integer values

MOMIX: a decision space search method for (MOMIC)

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Some notation

By B^g, $B^{\mathbb{Z}}$ and $B^{g,\mathbb{Z}}$ we denote the following sets related to the constraints in (MOMIC):

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Using these sets, we can write (MOMIC) in short form as

$$\min\ f(x)$$

$$\text{s.t. } x \in B^{g,\mathbb{Z}}$$

Upper Bounds and Local Upper Bounds

Two lists of points are kept updated and used for pruning:

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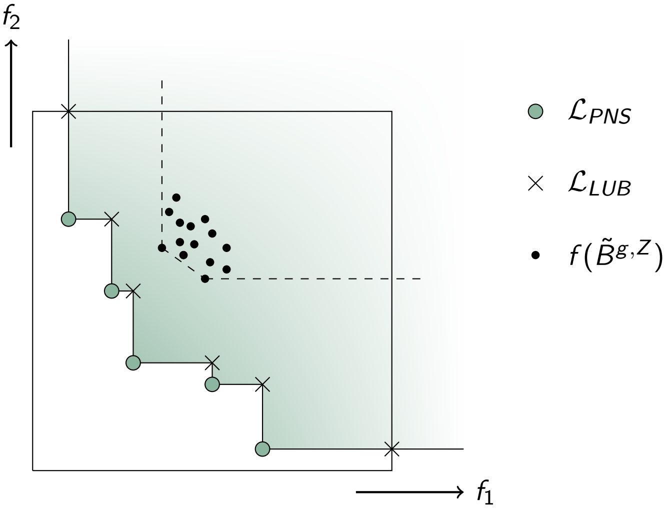
$$\textit{If} \quad p \notin f(\tilde{B}^{g,\mathbb{Z}}) + \mathbb{R}^m_+ \quad \textit{holds for all } p \in \mathcal{L}_{LUB}$$

$\tilde{B}$ does not contain any efficient point for (MOMIC)

Pruning of the node

example on a bi-objective purely integer instance



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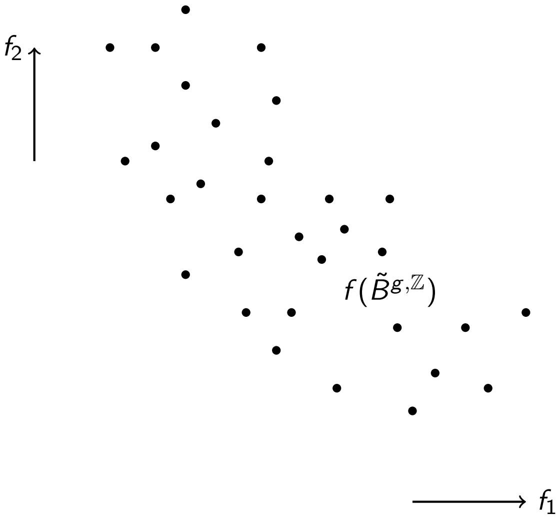

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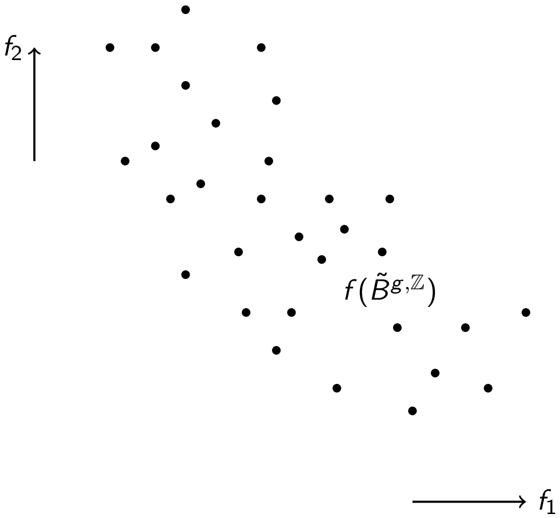

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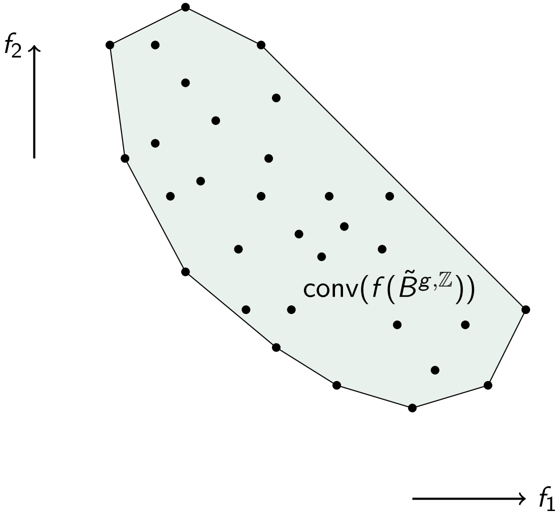
a lower bound is any set $L_{\tilde{B}} \subseteq \mathbb{R}^m$ such that

$$f(\tilde{B}^{g,\mathbb{Z}}) \subseteq L_{\tilde{B}} + \mathbb{R}^m_+$$

Lower bounds

convex hull of the image set



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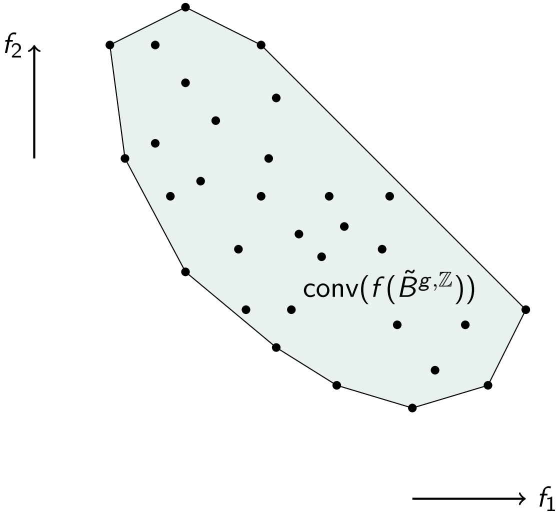

In particular **$\mathbf{conv}(f(\tilde{B}^{g,\mathbb{Z}}))$ is a lower bound**

we look for sets $L_{\tilde{B}}$:

$$\mathrm{conv}(f(\tilde{B}^{g,\mathbb{Z}})) \subseteq L_{\tilde{B}} + \mathbb{R}^m_+$$

Lower bounds computation

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$$\Downarrow$$

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Lower bounding procedure: Step 1

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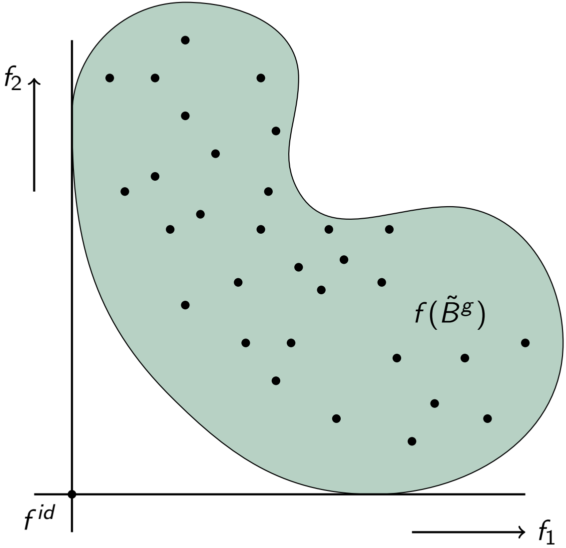

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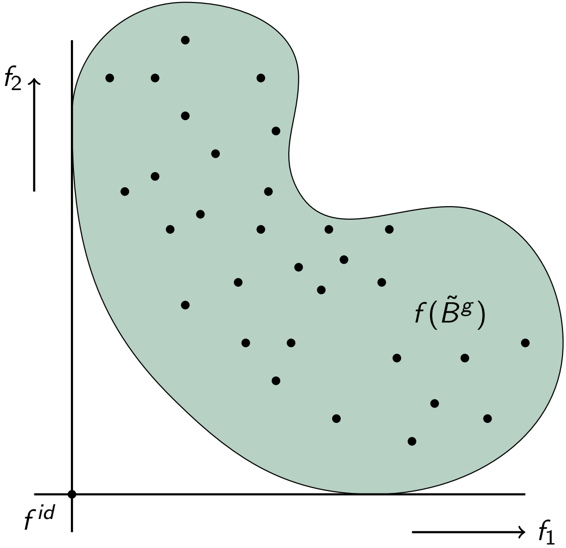

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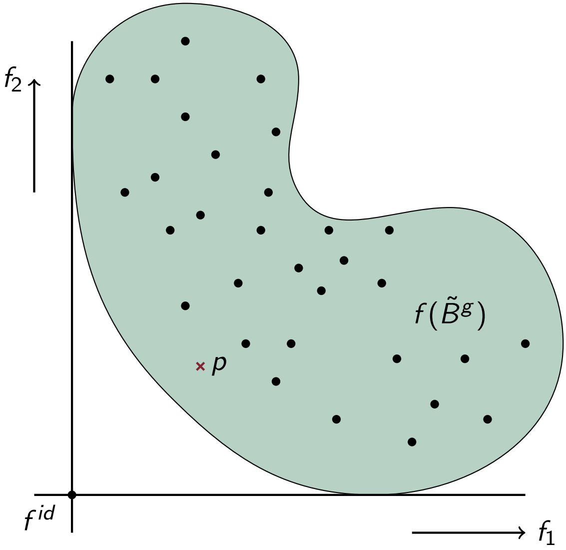
$$f_j^{id} := \min_{x \in \tilde{B}^g} f_j(x)$$

$$j = 1, \ldots, m$$

Lower bounding procedure: Step 2

computation of supporting hyperplanes for $f(\tilde{B}^g)$



Let $p \in \mathcal{L}_{LUB}$

if $p \in L_{\tilde{B}} + \mathbb{R}^m_+$ we try to **improve** $L_{\tilde{B}}$ by **computing a further hyperplane**

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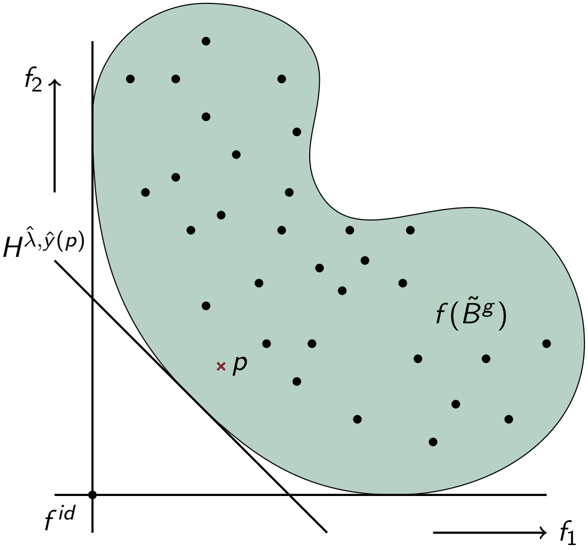

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Computation of supporting hyperplanes for $f(\tilde{B}^g)$

address a single-objective continuous convex problem

Let $(\hat{x}, \hat{t}) \in \tilde{B}^g \times \mathbb{R}$ be a minimal solution of the problem

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Then a supporting hyperplane of $f(\tilde{B}^g)$ is given by

$$H^{\hat{\lambda}, \hat{y}(p)} := \{y \in \mathbb{R}^m \mid \hat{\lambda}^T y = \hat{\lambda}^T \hat{y}(p)\}$$

with

- $\hat{\lambda} \in \mathbb{R}^m_+$ a Lagrange multiplier for $f(\hat{x}) \leq p + \hat{t}e$
- $\hat{y}(p) := p + \hat{t}e$

see e.g. [Löhne et al., J. Global Optim. (2014)]

Computation of supporting hyperplanes for $f(\tilde{B}^g)$

Implications

There exist two possibilities:

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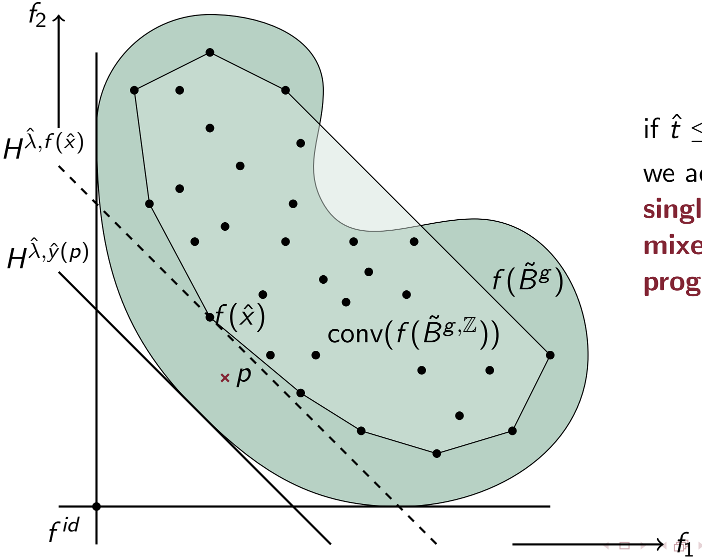
we cannot prune the node

we refine the outer approximation of $conv(f(\tilde{B}^{g,\mathbb{Z}}))$

Lower bounding procedure: Step 3

computation of supporting hyperplanes for $conv(f(\tilde{B}^{g,\mathbb{Z}}))$



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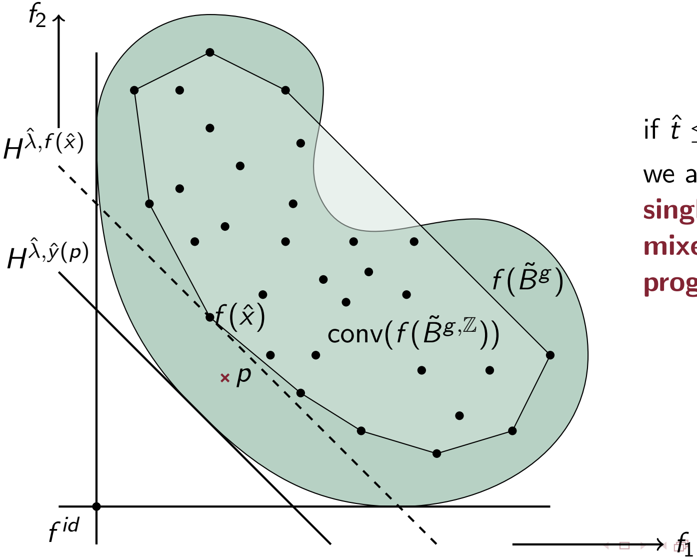

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$$\min \hat{\lambda}^T f(x)$$
$$\text{s.t. } x \in \tilde{B}^{g,\mathbb{Z}}$$

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and we branch the current node by bisecting $\tilde{B}$

Correctness results

detection of both the efficient and the nondominated set

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Let $E \subseteq B^{g,\mathbb{Z}}$ be the efficient set of (MOMIC).
Let $\mathcal{L}_\mathcal{S}$ be the output of `MOMIX`. Then $\mathcal{L}_\mathcal{S}$ is a cover of E, namely

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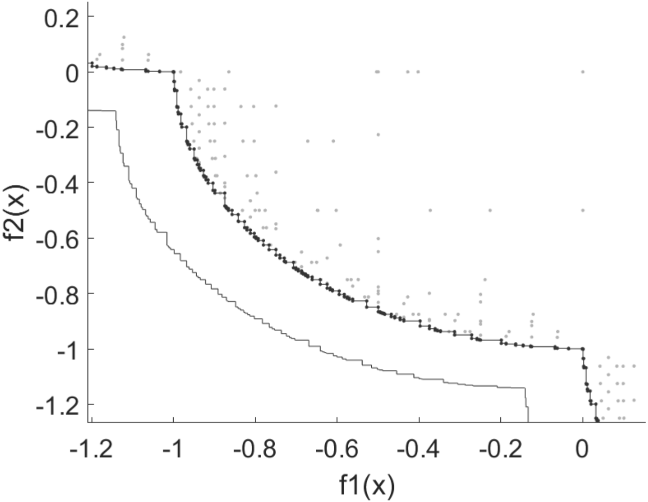
Theorem

Let $\delta > 0$ be the input parameter and $\mathcal{L}_{PNS}$, $\mathcal{L}_\mathcal{S}$ be the output of `MOMIX`. *Let $\mathcal{L}_{LUB}$ be the local upper bound set with respect to $\mathcal{L}_{PNS}$. Then*

$$f(E) \subseteq \Big(\bigcup_{p \in \mathcal{L}_{LUB}} (\{p\} - \mathbb{R}^m_+) \Big) \bigcap \Big(\bigcup_{z \in \mathcal{L}_{PNS}} (\{z - L\delta e\} + \mathbb{R}^m_+) \Big)$$

Example - bi-objective instance with $L\delta = 0.1\sqrt{2}$

part of the image set



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Choose $\hat{\imath} \in J_1$

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(br2) $J_2 = \operatorname{argmax}\{\tilde{u}_i - \tilde{l}_i \mid i \in \{1, ..., n\}\}$
If $J_2 \cap I \neq \emptyset$ holds, choose $\hat{\imath} \in J_2 \cap I$
Otherwise, choose $\hat{\imath} \in J_2$

Numerical results

Comparison between MOMIX and MOMIX$_{light}$

		MOMIX				MOMIX$_{light}$			
		(br1)		(br2)		(br1)		(br2)	
$\|I\|$	$\|C\|$	CPU	#nod	CPU	#nod	CPU	#nod	CPU	#nod
Test instance T2 - time limit 1800s									
1	2	40.1	757	38.7	765	849.9	609	524.5	669
2	2	30.8	537	31.6	575	667.2	555	563.0	641
3	2	31.0	535	30.8	521	1381.2	1127	814.4	917
4	2	34.7	567	65.6	1095	-	-	1134.9	1285
5	2	38.5	587	81.5	1259	-	-	-	-
10	2	350.3	2707	-	-	-	-	-	-
Test instance T3 - time limit 1800s									
1	2	15.5	301	14.6	299	1045.4	299	1025.6	299
10	2	36.5	413	27.1	353	-	-	-	-
20	2	-	-	46.9	411	-	-	-	-
30	2	-	-	80.4	471	-	-	-	-
50	2	-	-	-	-	-	-	-	-
Test instance T4 - time limit 3600s									
1	2	41.5	749	44.3	771	296.3	747	225.6	801
2	2	226.2	3683	240.5	3761	-	-	3090.4	3701
3	2	1354.9	19127	1321.5	18451	-	-	-	-
1	4	2199.5	23935	2246.6	24399	-	-	-	-

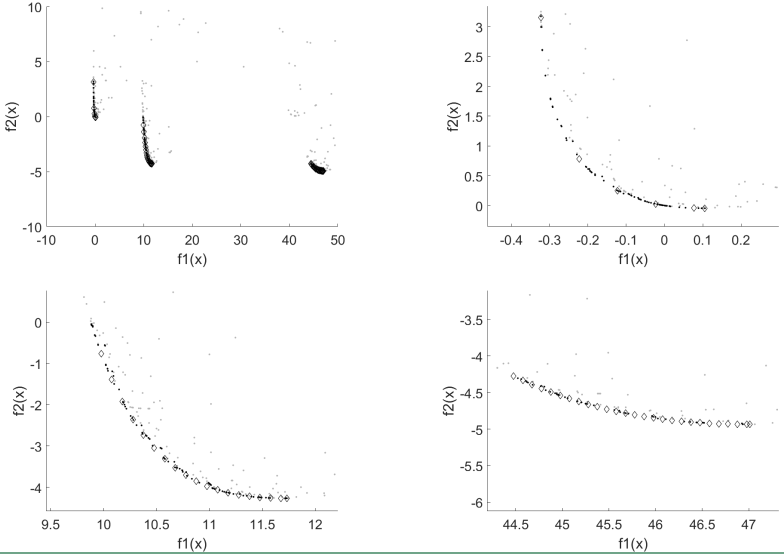
The ε-constraint method minimizes a sequence of parameter-dependent single-objective optimization problems of the following form:

$$\begin{aligned} \min \quad & f_2(x) \\ \text{s.t.} \quad & f_1(x) \leq \varepsilon \\ & x \in B^{g,\mathbb{Z}} \end{aligned} \tag{P$_\varepsilon$}$$

The minima of the functions f_1 and f_2 define the interval where the parameter ε belongs

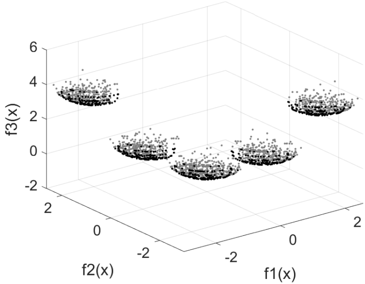
Comparison with the ε-constraint method

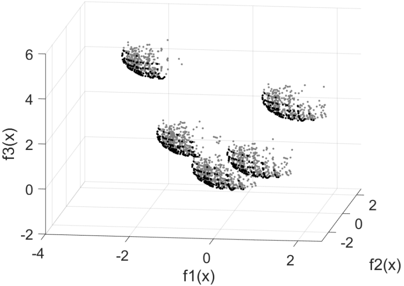
Instance T2 with $|I| = 5$, $n = 7$: $\mathcal{L}_{PNS}$ vs 52 solutions ($\diamond$) computed by ε-constraint method, solving 475 single-objective mixed integer problems



Results on a tri-objective instance

The set $\mathcal{L}_{PNS}$ from two different perspectives





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- correctness guarantee in terms of detecting both the efficient and the nondominated set of multiobjective mixed integer convex problems according to a prescribed precision

Thanks for your attention!

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